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Open strings, holography and stochastic processes

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Abstract: We use the correspondence between string states and local operators on the world-sheet boundary defined by vertex operators in open string theory to put in correspondence, holographically, the bosonic open string with the large-N limit of a mechanical system living on the world-sheet boundary. We give a natural interpretation of this system in terms of a one-dimensional stochastic process and show that the correspondence takes the form of a map between two conformal field theories with central charge $c = 24$ and $c = 1$.

Keywords: Bosonic Strings, AdS-CFT Correspondance, Stochastic Processes.

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Contents

1. Introduction

Hundred years ago the german physicist Max Plank solved the problem of black body radiation, using an idea that can be reinterpreted in view of the recently proposed holographic principle. He started with a three-dimensional (3D) ensemble of harmonic oscillators affected by a trivial ultraviolet divergence. Then, introducing a fundamental scale, he regularized the theory and ended up with a four-dimensional (4D) statistical field theory.

Recent exiting developments in high energy physics, in particular the holographic principle and the anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence [1], suggest that a lower dimensional version of the Planck mechanism may be at work also in this context. In fact, it has been shown that conformal field theories (CFT) in $d = 2$ can be put in correspondence with the large-N limit of a mechanical system. More in detail, investigations of the AdS/CFT correspondence in two dimensions [2] show that two-dimensional (2D) gravity can be described in terms of an ensemble of simple mechanical systems (representing a De Alfaro-Fubini-Furlan (DFF) [3] model) [4]. Another piece of evidence in this direction is the large-N limit of Calogero models, which can be shown to describe a CFT in $d = 2$ [5] (The large-N limit we consider in this paper should not be confused with the usual thermodynamic limit, where the particle density is kept constant).

One may therefore conjecture that the Plank idea represents the simplest realization of a more general principle, which has holographic nature and relates the large-N limit of mechanical systems with field theories (in particular CFT and gravity) in $d \geq 2$.

In this paper we test this conjecture in a simple context, namely open strings with 2D target space. The vertex operators in string theory define a natural, holographic, correspondence between string states and local operators on the world-sheet boundary. We use this correspondence to relate the open string theory with the large-N limit of a mechanical system living on the world-sheet boundary.

Since it is known that Wiener processes may be realized in terms of an ensemble (or if you want a large-N limit) of independent normal modes [6], it turns out that the most natural framework to describe our correspondence is to consider stochastic processes in $d = 1$. We will show that the correspondence takes the form of a map between two CFT's. Moreover, the mechanical system has a natural interpretation in terms of an ensemble of decoupled harmonic oscillators. To regularize the short distance behavior of the system we have to resume the hundred-years old Plank regularization procedure.

The structure of the paper is as follows. In section 2 we interpret the vertex operators in open string theory as a holographic correspondence. In section 3 we use them to construct the physical spectrum of open string theory in $D = 2$. In section 4 we consider a one-dimensional (1D) stochastic process and show that it can be put in correspondence, both at level of spectrum and Hilbert space, with open string theory. Finally in section 5 we state our conclusions.

2. Open strings, vertex operators and the holographic principle

In string theory there is nice correspondence between string states and local operators (see for instance ref. [7]). This correspondence is realized explicitly using the vertex operators associated with the string states. Although this fact is a general feature of string theory, for open strings it becomes a genuine realization of the holographic principle in two spacetime dimensions. In the case of open strings the world-sheet, whose coordinates are τ and σ, has a timelike boundary (the line swept out by the strings endpoints at $\sigma = 0, \pi$) so that the correspondence takes the form of an isomorphism between string states on the two-dimensional "bulk" and local operators on the one-dimensional boundary.

In the following will focus on the bosonic open string with 2D target space. Since the above correspondence does not depend on the target space dimension, our idea is expected to work in any dimension. We impose Neumann boundary conditions $(n^a \partial_a X^\mu(\tau,\sigma)|_{\sigma=0,\pi} = 0)$. The string field X^μ has the normal mode expansion (we will follow the conventions of ref. [8])

$$X^\mu(\tau,\sigma) = x^\mu + p^\mu \tau + i \sum_{n \neq 0} \frac{\alpha_n^\mu}{n} e^{-in\tau} \cos(n\sigma) \,, \tag{2.1}$$

where $\mu = 0, 1$, $-\infty \leq \tau \leq \infty$ and $0 \leq \sigma \leq \pi$.

We will show that, although very simple, the case of a 2D target space is physically rich enough to give a non trivial realization of the holographic principle. Moreover, it has been shown that the AdS/CFT correspondence in $d = 2$ can be realized in terms of a 2D bosonic string with Neumann or Dirichlet boundary conditions [9]. Therefore, the string-states/local-operators isomorphism discussed here can be also considered as an explicit realization of the AdS_2/CFT_1 correspondence.

The explicit form of the string states/local-operators correspondence can be easily worked out. Passing to complex world-sheet coordinates $z, \bar{z}$ and expressing α^μ_n as contour integral in the complex plane one gets [7]

$$\alpha^\mu_{-m} = \frac{2i}{(m-1)!}\partial^m X^\mu(0,0)\,.$$

This equation defines a correspondence between string states and (normal-ordered) local operators $\mathcal{A}$ at the origin of the complex plane. For instance we have for the vertex operators,

$$|0,p\rangle \cong :e^{ip_\mu X^\mu(0,0)}:\,. \tag{2.2}$$

Considering time evolution, generated by the string hamiltonian H,

$$\mathcal{A}(\tau) = e^{i\tau H}\mathcal{A}(0,0)e^{-i\tau H}\,,$$

the correspondence holds for local operators $\mathcal{A}(\tau)$, so that we are dealing with an holographic correspondence between 2D bulk string states and local operators evaluated on the 1D timelike boundary $\sigma = 0, \pi$. If $\mathcal{A}(\tau)$ has conformal dimension J we have

$$[L_m, \mathcal{A}(\tau)] = e^{im\tau}\left(-i\frac{d}{d\tau} + mJ\right)\mathcal{A}(\tau)\,, \tag{2.3}$$

where L_m are the Virasoro generators. When applied to the vertex operator $\exp(ip_\mu X^\mu)$ of eq. (2.2), the previous equation gives a relation between conformal dimension of the operator and mass M of the string state [8]: $J = -M^2/2$. This equation represents the two-dimensional version of the general relation, found in the context of the AdS/CFT correspondence, between dimensions of operators of the conformal field theory and masses of states of AdS-gravity [10].

In the next sections we will explore the physical content of the holographic correspondence between 2D open string states and local operators on the world-sheet boundary.

3. Vertex operators and the string spectrum

At first sight the 2D bosonic string theory seems rather trivial: because there are no transverse directions, all the local degrees of freedom can be gauged away. A closer investigation reveals a topological obstruction that prevents this. For a $2D$ target

JHEP01(2001)007

space it is impossible to completely fix the reparametrization and Weyl scaling gauge symmetry and to impose Neumann boundary conditions avoiding at the same time that the 2D world-sheet degenerates into a line. On the other hand, it is evident that in case of a 2D target space we cannot resort to a non covariant formalism as in the $D > 2$ case (light-cone quantization for instance). In view of the holographic correspondence we are going to discuss, one may argue that the bulk gauge degrees of freedom become in some sense physical on the boundary.

In this section we aim to construct the Hilbert space for the bosonic open string with 2D target space. This can be easily done by considering it as a particular case of the general formulation of the bosonic string (see e.g. ref. [8]). It can be shown that the Hilbert space has a fairly simple structure: it is the Verma module of an infinite dimensional conformal algebra with central charge $c = 24$.

In order to fix the notations let us briefly review a few basics about string quantization. The normal modes α_n^μ appearing in eq. (2.1) are interpreted as operators obeying the usual quantization conditions: $[\alpha_n^\mu, \alpha_m^\nu] = n\delta_{m+n}\eta^{\mu\nu}$ and $[x^\mu, p^\nu] = i\eta^{\mu\nu}$. We will use light-cone coordinates in the target space: $\eta_{+-} = \eta_{-+} = -1$. The Hilbert space is spanned by the base of states given by

$$\prod_{\mu=0,1} \prod_{n=1}^{\infty} (\alpha_n^\mu)^{\epsilon_{n,\mu}} \left|0, p\right\rangle , \tag{3.1}$$

where $\epsilon_{n,\mu} = 0, 1, \ldots$ are the occupation numbers. The physical states are defined by the Virasoro conditions

$$(L_m - a\delta_m) \left|\phi\right\rangle = 0 \qquad m \geq 0, \tag{3.2}$$

where a is a c number, the conformal weight of the vacuum. We have to eliminate the unphysical ghosts (negative norm states) imposing the constraints (3.2) in a way consistent with the Lorentz invariance of the theory. In $D = 2$ we have the peculiarity that there are no effective normal modes. Indeed, it is well known [11] that the spectrum generating algebra for the 2D system is isomorphic to the conformal algebra generated by L_m. The physical spectrum can be found using the Brower construction [11], a version of the old-fashioned covariant formalism of Del Giudice, Di Vecchia and Fubini [12].

In this approach a set of operators that commute with the generators L_m are constructed. These operators acting on the ground state $\left|0, p\right\rangle$, give the physical states of the theory. It is important to choose properly the vacuum, that is to fix the highest weight state, $L_0 \left|a\right\rangle = a \left|a\right\rangle$. On general grounds we know that in order to obtain a ghost free spectrum, we must impose the condition $a \leq 1$.

The mass-shell condition fixes the ground state:

$$\begin{aligned} L_0 &= \alpha' p^2 + \sum_{n>0} \alpha_{-n} \cdot \alpha_n \\ (L_0 - a) \left|a\right\rangle &= 0 \Rightarrow \alpha' p^2 = a \,. \end{aligned}$$

JHEP01(2001)007

In the following we will set $\alpha' = 1/2$ and choose $p^+ = 1$ and $p^- = -a$. Note that the ground state is tachionic only for $0 < a \leq 1$. If we define k_n such that $k_n^+ = 0$, $k_n^- = -n$, then the vertex operators defined on the world-sheet boundary, $V(k_n, \tau) = \exp(ik_n \cdot X(\tau, 0))$, are periodic functions of τ with period 2π. We consider the vertex operators [11]

$$V^-(k_n,\tau) =: \dot{X}^- e^{inX^+} : -\frac{1}{2} in \frac{d}{d\tau} (\log \dot{X}^+) e^{inX^+} . \tag{3.3}$$

It is straightforward to check using eq. (2.3) that $V^-(k_n, \tau)$ has conformal dimension $J = 1$. The second term in eq. (3.3) is exactly what is needed to cancel the anomalous, k_n dependent, dimension due to the normal ordering in the first term. It follows that the operators

$$A_n^- = \frac{1}{2\pi} \int_0^{2\pi} d\tau\, V^-(k_n, \tau)$$

trivially commute with the Virasoro generators: $[L_m, A_n^-] = 0$ and can be used to define the physical states. With the redefinition $\tilde{A}_n^- = A_n^- + \delta_n$ we find the commutation relations:

$$[\tilde{A}_n^-, \tilde{A}_m^-] = (n-m)\tilde{A}_{n+m}^- + 2(n^3 - n)\delta_{n+m} . \tag{3.4}$$

This is a Virasoro algebra with central charge $c = 24$. It is easy to check that

$$\tilde{A}_n^- \left|0, p\right\rangle = 0 , \qquad n > 0 .$$

Moreover, since $\tilde{A}_0^- \left|0,p\right\rangle = (p^- + 1)\left|0,p\right\rangle = (1-a)\left|0,p\right\rangle$, we find that $\left|0,p\right\rangle$ is a highest weight state for the algebra (3.4) with weight $h = 1 - a$. The case $a = 1$ will be excluded since we want to avoid a zero norm ground state.

It is a corollary of the no-ghost theorem in D dimensions that physical states in our case are given by the Verma module $V(c, h)$, representing the physical states of the full Hilbert space (3.1), with $c = 24$ and $h = 1 - a$,

$$V(24, 1-a) = \{\tilde{A}_{-n_1}^- \tilde{A}_{-n_2}^- \cdots \tilde{A}_{-n_k}^- \left|0,p\right\rangle \mid 1 \leq n_1 \leq \cdots \leq n_k; k > 0\} . \tag{3.5}$$

All the states (3.5) have the same energy: $L_0 \left|\phi\right\rangle = a \left|\phi\right\rangle$, $\forall \left|\phi\right\rangle \in V(24, 1-a)$, and are annihilated by the conformal generators L_m for $m > 0$. It is important to observe that the module $V(24, 1-a)$ does not contain any null submodule, so that it gives an irreducible representation of the Virasoro algebra: $V(24, 1-a) = M(24, 1-a)$, $M(c,h)$ being the irreducible representation obtained quozienting out of the Verma module $V(c,h)$ the null submodules. Indeed, the Kac determinant never vanishes in the region $c > 1$ and $h > 0$ (see for instance ref. [13]). This means that all the states at a given level of the Verma module $V(24, 1-a)$ are linearly independent.

4. One-dimensional stochastic processes and holography

In the previous section we have identified precisely the physical Hilbert space of the 2D bosonic string. In this section we will try to realize the holographic idea. We will look for a mechanical system, defined on the timelike boundary of the open string world-sheet, whose Hilbert space can be mapped exactly on the physical Hilbert spaces of the string (3.5).

Trying to setup this correspondence one can hardly escape a basic problem: how can a one-dimensional system have the same degeneracy of a 2D field theory? To be more concrete let us consider the simple example of DFF conformal quantum mechanics [3]. Since the compact operator R, basically the hamiltonian, of ref. [3] has the same spectrum of the harmonic oscillator, $r_n = r_0 + n$, where r_0 is a constant depending on the coupling and $n \geq 0$, neglecting the zero point term the free energy is given by $F = k_B T \ln(1 - e^{-\beta})$, where $\beta = 1/k_B T$. On the other hand, for a 2D field theory we expect a Stefan-Boltzmann behavior $F \propto T^2$ and an entropy $S \propto T$. This fact indicates that in $d = 1 + 0$, an interaction between a finite number of particles cannot give the required degeneracy. A possibility left open is to consider the limit $N \to \infty$ of a system of N interacting particles, whose interaction is possibly described by a Calogero model (see for instance ref. [14]).

In this paper we explore a situation similar to that described in ref. [4]. In that paper it was shown that the quadratic dependence of the free-energy from the temperature may emerge from the coupling of the DFF model with an arbitrary time-dependent external source. Along this line of thoughts it seems natural to consider stochastic processes.

Our starting point is a simple idea based on a well-known property of the Wiener process: if $\alpha_0, \alpha_1, \ldots$ are a sequence of gaussian variables, each with the same distribution $(2\pi)^{-1/2} \exp -\alpha^2/2$, then the coordinate $y(t)$ of the stochastic process in $d = 1$ may be written as

$$y(t) = \frac{t}{\sqrt{\pi}}\alpha_0 + \left(\frac{2}{\pi}\right)^{1/2} \sum_{n>0} \frac{\sin nt}{n} \alpha_n \,. \tag{4.1}$$

Indeed a gaussian process is completely determined by its covariance and the covariance of the random function $y(t)$, inferred from the distribution of the α, is the same as the covariance of a Wiener process for $0 \leq t \leq \pi$, with sample paths in one dimension and diffusion coefficient equal to 1/2 [6]. Upon quantization, on the basis of the spectrum given by the α modes it is conceivable to extract, under certain conditions, an infinite dimensional conformal algebra, or what is the same, to define a mapping between the Hilbert space of this system and the one of the 2D open string theory described in the previous section.

It is worth observing that although (4.1) has a close resemblance with the open string mode expansion (2.1) evaluated on the $\sigma = 0$ boundary, the correspondence between the 1D and 2D theories is not immediate. In particular the normal modes of the string in eq. (2.1) cannot be identified with the modes α appearing in eq. (4.1). The crucial point is that in the string case we have the Lorentz (causal) structure, typical of any local field theory, which induces, upon quantization, negative norm states. In the physical theory arising after the eliminations of the ghosts from the spectrum, there are no more normal modes but only Virasoro operators generating a Verma module. On the other hand the 1D stochastic process does not know anything about Lorentz invariance, so that the normal modes α appearing in eq. (4.1) generate directly the physical spectrum. For this reason the correspondence between the two theories can be made only at the level of the associated physical spectra and physical Hilbert spaces.

Basically to each path $y(t)$, defined by eq. (4.1) in a finite time interval $[0, L]$ we associate an energy

$$E[y] = \frac{1}{2}\int_0^L dt\,\left(\frac{dy}{dt}\right)^2 . \tag{4.2}$$

Physically $E[y]$ may be interpreted as the energy dissipated in a thermal reservoir by a random perturbation with a finite temporal extension. Of course the most natural interpretation would be to think of E as the energy loss by a brownian particle driven by some external force, but in this case the temporal boundary conditions should be chosen differently. If we consider t as a space dimension, instead of a temporal one, a simple mechanical interpretation is to think of E as the elastic energy of a weightless elastic string clamped at the endpoints [15].

In a statistical treatment to each path is associated a probability factor

$$e^{-\beta E} = \exp\left\{-\frac{1}{2}\beta\int_0^L dt\,\left(\frac{dy}{dt}\right)^2\right\} . \tag{4.3}$$

Discretizing the energy integral it is easy to define a stochastic process $Y(t)$ whose sample functions are the paths $y(t)$ [15]. The stochastic process may be described as follows. If we choose n different points in the interval $(0, L)$ and label them with increasing time

$$0 < t_1 < t_2 < \cdots < t_n < L\,, \tag{4.4}$$

the energy of a piece-wise straight path is given by

$$\frac{1}{2}\sum_{k=0}^{n}\frac{(y_{n+1}-y_n)^2}{t_{n+1}-t_n},$$

where $t_0 = 0$, $t_{n+1} = L$, $y_0 = y_{n+1} = 0$. A stochastic process $Y(t)$ is completely specified once is defined a hierarchy of functions $P_n(y_1, t_1; y_2, t_2; \ldots ; y_n, t_n)$, $n \in \mathbb{N}$, giving the joint probability densities that Y has the value y_1 at the time t_1, y_2 at the time t_2 and so on. From P_n we compute the correlations:

$$\langle Y(t_1)Y(t_2)\cdots Y(t_n)\rangle = \int y_1 y_2 \cdots y_n\, P_n(y_1, t_1; y_2, t_2; \ldots ; y_n, t_n)\, dy_1 dy_2 \cdots dy_n\,.$$

In order to specify a stochastic process the hierarchy of probability distributions has to obey the four consistency conditions (Kolmogorov)

$$
\begin{aligned}
&\text{(i)} \quad P_n \geq 0\,; \\
&\text{(ii)} \quad P_n \text{ is a symmetric function of the pairs } (y_k, t_k)\,; \\
&\text{(iii)} \quad \int P_n(y_1, t_1; y_2, t_2; \dots; y_n, t_n)\, dy_n = P_{n-1}(y_1, t_1; y_2, t_2; \dots; y_{n-1}, t_{n-1}) \\
&\text{(iv)} \quad \int P_1(y_1, t_1)\, dy_1 = 1\,.
\end{aligned}
\tag{4.5}
$$

Given the time-ordering (4.4), we define the hierarchy satisfying the conditions (4.5) by setting,

$$
\begin{aligned}
P_n(y_1, t_1; y_2, t_2; \dots; y_n, t_n) = & \left(\frac{2\pi L}{\beta}\right)^{1/2} \prod_{k=0}^{n} \left(\frac{\beta}{2\pi(t_{k+1} - t_k)}\right)^{1/2} \times \\
& \times \exp\left[-\frac{\beta}{2}\frac{(y_{n+1} - y_n)^2}{t_{n+1} - t_n}\right],
\end{aligned}
\tag{4.6}
$$

which in the $n \to \infty$ limit defines the functional integral measure with weight given by eq. (4.3).

The probability distributions P_n define a gaussian but not stationary stochastic process. The correlation functions are readily computed: $\langle Y(t) \rangle = 0$ whereas if $t_1 \leq t_2$ the autocorrelation is

$$
\langle Y(t_1) Y(t_2) \rangle = \frac{1}{\beta} \frac{t_1(L - t_2)}{L} \qquad t_1 \leq t_2\,.
$$

It should be noted that the autocorrelation does not depend on $|t_1 - t_2|$ alone since the process is not stationary. We also observe that due to the presence of L, which has the physical meaning of an infrared cutoff (see below, eq. (4.9)) the process (4.6) is not markovian. Nevertheless for a set of n successive times $t_1 < t_2 < \cdots < t_n < L$ one verifies the property

$$
P_{1|n-1}(y_n, t_n | y_1, t_1; y_2, t_2; \dots; y_{n-1}, t_{n-1}) = P_{1|1}(y_n, t_n | y_{n-1}, t_{n-1})\,,
$$

where $P_{l|k}(y_{k+1}, t_{k+1}; \dots; y_{k+l}, t_{k+l} | y_1, t_1; y_2, t_2; \dots; y_k, t_k)$ is the conditional probability of observing y_{k+1} at the time t_{k+1} etc. given the k events y_1 at the time t_1 etc. Our process is simply related with the Wiener-Lévy process [15], defined by

$$
\begin{aligned}
P_1(y_1, 0) &= \delta(y_1) \\
P_{1|1}(y_2, t_2 | y_1, t_1) &= \left(\frac{\beta}{2\pi(t_2 - t_1)}\right)^{1/2} \exp\left[-\frac{\beta}{2}\frac{(y_2 - y_1)^2}{(t_2 - t_1)}\right],
\end{aligned}
\tag{4.7}
$$

where $t_2 > t_1$ and $-\infty < y < \infty$. It turns out that (4.7) is simply the limit $L \to \infty$ of the process (4.6).

Once we have specified unambiguously the stochastic process we are dealing with, we turn to the calculation of the mean energy of our system. Given the boundary conditions $Y(0) = Y(L) = 0$ we expand Y in Fourier modes as

$$Y(t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{L} t\right).$$

The correlations for the coefficients A_n are obtained from the correlations for the process $Y(t)$. We get

$$\begin{aligned} \langle A_n \rangle &= 0 \\ \langle A_n A_m \rangle &= \delta_{n-m} \frac{2L}{\beta n^2 \pi^2}. \end{aligned} \tag{4.8}$$

In order to compute the mean energy we have to quantize the system: the problem of the mean energy is of course the celebrated problem of the black-body radiation solved by Planck. We interpret the A_n as decoupled quantum harmonic oscillators. We simply apply the Planck's law changing in the l.h.s. of eq. (4.8) the frequency distribution, i.e. we perform the replacement:

$$\frac{1}{\beta} \to \frac{n\pi/L}{e^{\beta n\pi/L} - 1}$$

(we use units where $\hbar = 1$). Hence for the mean energy we have

$$\begin{aligned} \langle E \rangle &= \int \mathcal{D}y\, E[y] e^{-\beta E[y]} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} \langle A_n^2 \rangle \\ &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} \frac{2L}{n^2\pi^2} \frac{n\pi/L}{e^{\beta n\pi/L} - 1} \to \frac{L}{2\pi} \int_0^{\infty} d\omega \frac{\omega}{e^{\beta\omega} - 1}. \end{aligned}$$

From this equation it follows immediately,

$$\langle E \rangle \equiv U = \frac{\pi L}{12} (k_B T)^2. \tag{4.9}$$

Because, as we shall see in the following, the system defines a CFT, from eq. (4.9) we can read out the central charge, $c = 1$ (see for instance ref. [16]). For the statistical entropy S at the temperature T we find

$$S = \frac{\pi L}{12} k_B^2 T.$$

4.1 The mapping between the Hilbert spaces

Until now we have worked out the correspondence between 2D bosonic string and stochastic process at the level of the spectra of the two theories. In particular, it follows that the two theories have the same partition function. Let us now give simple arguments to show that the correspondence holds also for the physical Hilbert spaces of the two theories. In a formal quantum treatment the 1D stochastic model we considered is described by the means of an Hilbert space whose structure is given by the familiar Fock space generated by creation α_{-n} and annihilation α_n ($n > 0$, $n \in \mathbb{N}$) operators, with commutation relations $[\alpha_n, \alpha_m] = n\delta_{m+n}$, acting on a vacuum $|0\rangle$. But this space can be exactly mapped on the Verma module (3.5). Since the Verma module does not contain null submodules, the one-to-one correspondence between the respective states follows immediately.

4.2 Conformal symmetry and stochastic processes

Eq. (4.2) gives the simplest realization of our idea. Of course there are many other possibilities. An amusing example may be the following. Let $I_\nu(z)$ be the Bessel function with imaginary argument and $\mu = \sqrt{g^2 + 1/4}$, where $g \in \mathbb{R}$. For $y_0 > 0$ and $0 < y_i < \infty$, $i = 1, \ldots, n$; $y_{n+1} = y_0$, given the time ordering (4.4) with $t_0 = t_{n+1} = 0$, we define the process

$$P_n(y_1, t_1; y_2, t_2; \ldots ; y_n, t_n) = \left(e^{-\frac{\beta y_0^2}{L}}\, y_0\, I_\mu\left(\frac{\beta y_0^2}{L}\right)\right)^{-1} \frac{L}{\beta}\frac{\beta}{t_1}\frac{\beta}{t_2 - t_1} \cdots \frac{\beta}{L - t_n} \times$$
$$\times \prod_{j=1}^{n+1} \left[\sqrt{y_j y_{j-1}} I_\mu\left(\frac{\beta y_j y_{j-1}}{t_j - t_{j-1}}\right)\right]^{1/2} \exp\left[-\frac{\beta}{2}\frac{(y_{n+1} - y_n)^2}{t_{n+1} - t_n}\right].$$

If for a different ordering the process is again defined by the symmetry condition (ii) of eqs. (4.5), then all the Kolmogorov axioms are obeyed. Again we have a non stationary and non markovian stochastic process, defined is such a way that (see ref. [17]), considering the limit $n \to \infty$, we get a functional integral measure with weight $e^{-\beta E[y]}$, where now $E[y]$ is given by

$$E[y] = \frac{1}{2}\int_0^L dt \left[\left(\frac{dy}{dt}\right)^2 + \frac{g^2}{y^2}\right].$$

This establishes a link between conformal quantum mechanics [3] and stochastic process. It is curious to observe that we can define a stochastic process only if in the functional integration we have a conformally symmetric weight. If a dimensional parameter is present, say a mass, then there is no way to satisfy the crucial Martingale property (iii) of the Kolmogorov axioms (4.5).

5. Summary and outlook

In this paper we have shown that the map between string states and local operators defined by the vertex operators in bosonic opens string theory, can be used to put in correspondence, holographically, open string theory with stochastic processes in one dimension.

The model we have described in this paper is too simple to be considered more than a toy model. However, it can be used to test general ideas. In particular it sheds light on the physical meaning of the AdS/CFT correspondence in two dimensions, or more in general on the holographic principle. The large-N limit of a mechanical system in one spatial dimension, with short distance behavior regularized by quantum mechanics, produces a field theory in 1+1 dimensions. Hence this mechanism can be used to generate one more dimension and a field theory out of an ensemble of particles. Here this is achieved in a non trivial way because we are going from a simple mechanical system to a two-dimensional local field theory (the bosonic open string), with a rich Lorentz and gauge structure.

Presently we do not know whether this mechanism is relevant also for the AdS/CFT correspondence and the holographic principle in $d > 2$. Intuitively, it seems to be a basic and universal mechanism that could work in general cases.

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